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Steady Shock Profile in a One-Dimensional Lattice

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The equations of motion of a one-dimensional lattice of mass points connected by nonlinear springs are set forth and compared with the equations of the corresponding continuum. A permanent regime for the damped lattice is obtained by series approximation and shown to agree with that of the continuum. A higher approximation leads to a permanent regime profile for the undamped lattice which oscillates steadily after shock arrival. This is shown to be in qualitative accord with the results of numerical integrations of the transient problem. However, comparison of periods of steady oscillation with those obtained in the transient problem indicate that the series approximation to the permanent regime is quantitatively unsatisfactory, though qualitatively correct. Scaling of the problem with a parameter $u_1\alpha$ is noted, where u_1 is steady particle velocity behind the shock and α is a parameter of nonlinearity.

I. INTRODUCTION

Considerable attention has been given to the discussion of steady shock-compression profiles in gases.[1] Much less work has been done on the analogous problem in solids, partly because a satisfactory microscopic model of a solid is not available, partly because the mathematics of nonlinear lattices is more formidable than that of random atomic assemblies, and partly because interest in shock waves in solids has generally tended to lag behind that in gases. Band[2] has discussed in general terms the steady profile problem. Bland[3] has obtained explicit profiles in a continuum for particular assumptions about the constitutive relations. These efforts are based principally upon continuum models of solids and require, as in gases, existence of time-dependent forces for definition of shock profiles.

When lattice models of solids are being considered, the processes for introducing dissipative mechanisms are less straightforward than for a continuum, since dissipation is now to be described in terms of irreversible relative motions of atoms which form the lattice or of their constituents. Anderson[4] has obtained steady profiles in a one-dimensional lattice with nonlinear forces by introducing dashpots in parallel with springs connecting atoms. It is shown in Sec. III of this paper that such a model leads to a smooth, non-oscillatory shock transition between two uniform states and that the transition is the analogue of that which occurs in the continuum, provided a certain expansion is properly truncated.

Numerical solutions of transient shock wave problems in lattices without dissipation have shown that even in such cases the shock profile has finite rise time and is oscillatory but not steady. The amplitude of oscillations behind the shock front decays with the passage of time

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[1] J. N. Bradley, *Shock Waves in Chemistry and Physics* (John Wiley & Sons, Inc., New York, 1962).
[2] W. Band, J. Geophys. Res. **65**, 695 (Feb. 1960).
[3] D. R. Bland, J. Inst. Math. Appl. **1**, 56 (1965).
[4] G. D. Anderson, Ph.D. thesis, Washington State University Pullman, Washington, 1964.

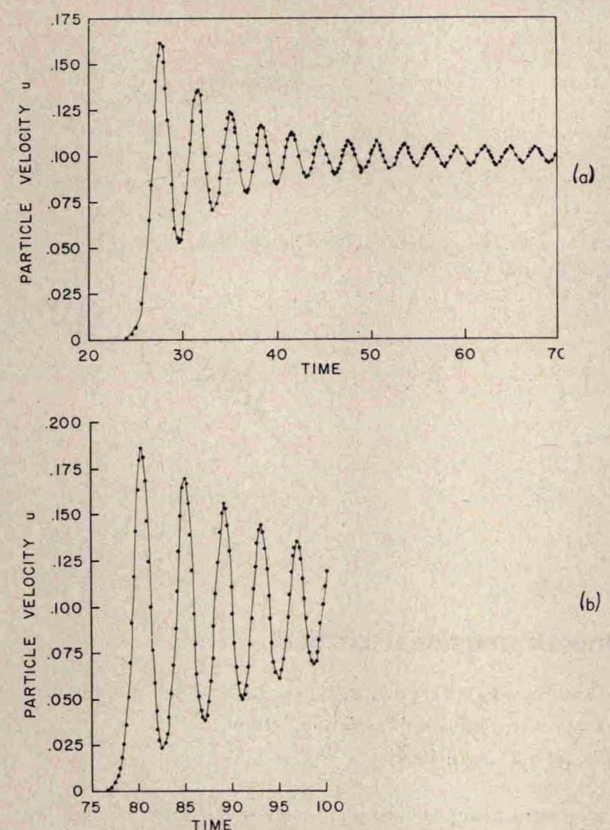

FIG. 1. Transient shock profiles from numerical integrations for semi-infinite lattice driven by step change in velocity: (a) 30 particles from driven end, (b) 90 particles from driven end.

because the lattice is dispersive.[5-7] Some typical results of such numerical integrations are shown in Fig. 1. Profiles of the kind shown there are disturbing for two reasons: (i) they are not steady, and all our experience in the continuum, which should be a limit of the lattice, indicates that steady profiles do exist, and (ii) the one nonlinear lattice problem which can be solved exactly,

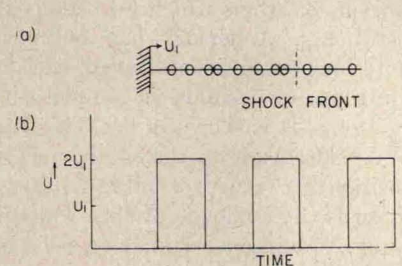

FIG. 2. Velocity profile for shock in a system of beads sliding on a wire: (a) bead positions and shock front at a particular time, (b) permanent regime profile for each bead.

[5] D. H. Tsai and C. W. Beckett, J. Geophys. Res. **71**, 2601 (15 May 1966).
[6] R. Manvi, G. E. Duvall, and S. C. Lowell, Int. J. Mech. Sci. **11**, 1 (1969).
[7] R. Manvi, "Shock Wave Propagation in a Dissipating Lattice Model," Ph.D. thesis, Department of Mechanical Engineering, Washington State University, Pullman, Washington, 1968.

viz. the sliding of perfectly elastic beads on a wire, as in Fig. 2, has each particle oscillating indefinitely with constant amplitude after the shock wave has passed. Such behavior constitutes, in the present context, a steady profile. A detailed examination of the more general problem of a one-dimensional lattice with nearest neighbor interaction and without dissipation is undertaken in Sec. IV. The mathematical problem posed is unusual, but an approximate permanent regime solution is found which is in harmony with the results shown in Figs. 1 and 2, though some disagreements between this solution and the transient case still exist.

II. EQUATIONS OF MOTION

The lattice model is illustrated in Fig. 3, including dissipative dashpots, as introduced by Anderson. The entire lattice is generated by translation of a single mass–spring–dashpot element, and mass points are constrained to move in the direction of the lattice. The separation between undisturbed masses is x_0. The sign convention used in describing forces is shown in Fig. 3. It is chosen opposite from that normally used because these forces will be compared with pressures, not stresses,

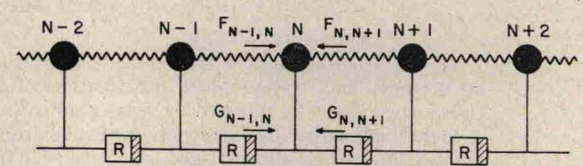

FIG. 3. Lattice model with damping.

in the continuum case. With this convention the force, $F_{N,N+1}$, exerted on mass N by $N+1$ is negative when the spring connecting N and $N+1$ is stretched beyond its equilibrium position. We assume the force to be nonlinear with parabolic form:

$$F_{N,N+1}=-(S_{N+1}-S_N)+\alpha(S_{N+1}-S_N)^2. \qquad (1)$$

Dimensionless variables are used here and in the equations following. The relative velocity of the two particles is assumed to generate a linear damping force:

$$G_{N,N+1}=-\eta(S_{N+1}'-S_N'), \qquad (2)$$

where $'\equiv d/dT$ and T is a dimensionless time. Combining Eqs. (1) and (2) with similar forces due to motion of the $N-1$ particle leads to an equation of motion:

$$S_N''(T)=-(F_{N,N+1}-F_{N-1,N})-(G_{N,N+1}-G_{N-1,N}). \qquad (3)$$

In order to pass to the continuum limit for uniaxial strain, we suppose that space is filled with parallel lattices like the one shown, one per unit area, and that $x_0=N\Delta x_0$ is a Lagrangian coordinate for the Nth particle. We then replace $F_{N,N+1}$ by $p(x_0+\Delta x_0/2)$, $F_{N-1,N}$ by $p(x_0-\Delta x_0/2)$, S_N by $S(x_0,T)$, and unit mass by $\rho_0\Delta x_0$. Then Eq. (3) becomes

$$\rho_0\Delta x_0 d^2S/dT^2=-p[(x_0+\Delta x_0)/2]+p[(x_0-\Delta x_0)/2]$$
$$+\eta[u(x_0+\Delta x_0)-2u(x_0)+u(x_0-\Delta x_0)],$$

where $u=dS/dT$. In the limit as $\Delta x_0\to 0$,

$$\frac{d^2S}{dT^2}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x_0}+\frac{\mu}{\rho_0}\frac{\partial^2 u}{\partial x_0{}^2}, \qquad (4)$$

where $\mu=\eta\Delta x_0$ is the ordinary viscosity and p is compressive stress in the x_0 direction. Equation (4) is the equation of motion of a continuum in uniaxial strain. In the dimensionless Eq. (3) the mass of each particle is unity and $\Delta x_0=1$, therefore, $\rho_0=1$, $\mu=\eta$, and x_0 is dimensionless.

To further relate the lattice under study to the continuum, we define as strain between the N and $N+1$ mass points $\epsilon_N=S_{N+1}-S_N$, since the undisturbed separation is taken as unity. The continuum analogue is $\epsilon=(V-V_0)/V_0$, where V_0 is undisturbed specific volume. Then Eqs. (1) and (2) become

$$-F_{N,N+1}=\epsilon_N-\alpha\epsilon_N{}^2$$
$$-G_{N,N+1}=\eta\epsilon_N'. \qquad (5)$$

Equations (5) go directly into the form assumed by Bland in calculating steady profiles[3]

$$-p-q=\sigma_x=\epsilon-\alpha\epsilon^2+\eta\epsilon'$$
$$-q=\eta\partial u/\partial x_0=\eta\epsilon'. \qquad (6)$$

In the continuum case Eq. (4) is supplemented by an equation of conservation of mass:

$$\partial x/\partial x_0=\rho_0/\rho=V, \qquad (7)$$

where x is the Euler coordinate. In the lattice problem mass conservation is assured by the assumption that each lattice point is occupied by a constant mass.

III. SHOCK PROFILE WITH DISSIPATION

The permanent regime or steady profile is obtained in the continuum case by solving Eqs. (4) and (7) subject to the conditions that $(\partial/\partial T)_x\equiv 0$, that a mass element approaches a uniform undisturbed state as $t\to-\infty$, and a uniform compressed state as $t\to+\infty$.[2] A condition equivalent to the first of these is that p, u, ρ be functions only of the variable $\xi=t-\theta x$, where θ is to be determined by the boundary conditions

$$du/d\xi\to 0,\ p\to p_0,\ u\to 0,\ \rho\to\rho_0 \quad \text{as} \quad \xi\to-\infty$$
$$du/d\xi\to 0,\ p\to p_1,\ u\to u_1,\ \rho\to\rho_1 \quad \text{as} \quad \xi\to+\infty. \qquad (8)$$

In either case we arrive at a relation

$$\theta^2\eta du/d\xi=(1-\theta^2)u-\alpha\theta^3u^2, \qquad (9)$$

where the term on the left represents the viscous force. The boundary conditions of Eq. (8) imply that

$$u_1=(1-\theta^2)/\alpha\theta^3. \qquad (10)$$

Equation (9) can be integrated directly, and if the origin, $\xi=0$, is chosen where $u=u_1/2$, we obtain

$$u=(u_1/2)[1+\tanh(u_1\alpha\theta\xi/2\eta)], \qquad (11)$$

where $U_s=1/\theta$ is shock velocity. Note particularly that when viscosity vanishes, the shock profile becomes a discontinuity in u, p, ρ, etc.

In the lattice problem we proceed in a similar manner. Combining Eqs. (1)–(3) produces the equation

$$S_N''(T)=(S_{N+1}-2S_N+S_{N-1})\{1-\alpha(S_{N+1}-S_{N-1})\}$$
$$+\eta(S_{N+1}'-2S_N'+S_{N-1}'), \quad (12)$$

where each displacement is evaluated at time T. We again seek progressing wave solutions in the form

$$S_N(T)=S(T-N\theta)\equiv S(\xi). \qquad (13)$$

If we define $D\equiv d/d\xi$, then

$$S_N'(T)=DS,$$
$$S_{N+1}(T)=S(\xi-\theta)=e^{-\theta D}S(\xi),$$
$$S_{N-1}(T)=S(\xi+\theta)=e^{\theta D}S(\xi). \qquad (14)$$

Equation (12) then becomes the ordinary differential equation,

$$D^2S=[2(\cosh\theta D-1)S][1+2\alpha(\sinh\theta D)S]$$
$$+\eta D[2(\cosh\theta D-1)S]. \quad (15)$$

By expanding the operators in series and keeping only the lowest order terms, we obtain the equation

$$D^2S=\theta^2D^2S+\theta^4D^4S/12+\cdots+2\alpha\theta^3DS\cdot D^2S+\cdots$$
$$+\eta\theta^2D^3S+\cdots. \quad (16)$$

If we discard terms of fourth order or higher and substitute $u\equiv DS$, $u'=Du$, etc, Eq. (16) becomes

$$(1-\theta^2)u'=2\alpha\theta^3uu'+\eta\theta^2u''. \qquad (17)$$

Integrating once

$$(1-\theta^2)u=\alpha\theta^3u^2+\eta\theta^2u'+A. \qquad (18)$$

Boundary conditions are the same as those in Eq. (8):

$$\xi\to-\infty,\ u'\to 0,\ u\to 0$$
$$\xi\to+\infty,\ u'\to 0,\ u\to u_1. \qquad (19)$$

Applied to Eq. (18), these yield the results $A=0$, $u_1=(1-\theta^2)/\alpha\theta^3$, as for the continuum case. Since Eqs. (9) and (18), with $A=0$, are identical with identical boundary conditions, we conclude that they lead to the same shock profile, Eq. (11), and to the same relation between shock and particle velocity, Eq. (10).

The present calculation has been carried out with a particular force law, Eq. (5), but we infer the following: In the presence of dissipation there is no distinction

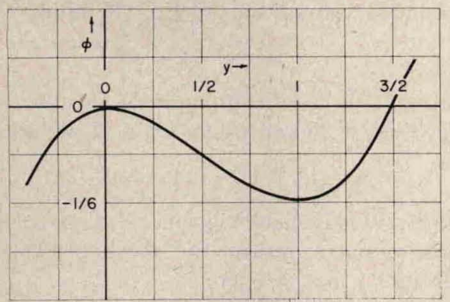

FIG. 4. Potential of Eq. (24).

TABLE II. Results of numerical integration of equation (12); $\eta=0$.

Computation number	α	u_1	$u_1\alpha$	θ	$(\theta/12\, u_1\alpha)^{1/2}$
P-26	0.1	0.1	0.01	0.99506	2.88
P-27	0.3	0.1	0.03	0.98554	1.66
P-28	1.0	0.1	0.1	0.95540	0.90
P-25	3.0	0.1	0.3	0.88856	0.50
P-32	10.0	0.1	1.0	0.75488	0.25

between the one-dimensional lattice and the continuum in uniaxial strain, except that the Lagrangian space variable x_0, is replaced by N, provided that higher-order derivatives in Eq. (16) are neglected.

IV. SHOCK PROFILE WITHOUT DISSIPATION

In the continuum, we noted that the shock transition becomes a discontinuity in the absence of rate-dependent forces. The same was found to be true of a damped lattice if the series of Eq. (16) was truncated at the third-order term. In both these cases, the shock transition is a smooth region of monotonic transition from the uncompressed initial state to the compressed final state. If, however, we include higher-order terms of Eq. (16), allowing η to become very small or vanish, a finite transition region remains. This can be illustrated by retaining the fourth-order term in Eq. (16). Then in place of Eq. (18) we have,

$$(1-\theta^2)u=\theta^4u''/12+\alpha\theta^3u^2+\eta\theta^2u'+A. \qquad (20)$$

The boundary conditions of Eq. (19) are now inadequate to evaluate A. We can, however, extend these boundary conditions on physical grounds. If we concede that any disturbance in a real lattice propagates at finite velocity, then a mass point ahead of the disturbance is not only at rest. It undergoes no acceleration until the disturbance reaches it; and in fact it is absolutely quiescent in the model we assume here, so that all derivatives vanish. This means that the series of Eq. (16) can be extended to arbitrarily high derivatives, and A will always vanish.

TABLE I. Amplitude dispersion of Eq. (25).

Amplitude a	ω Numerical integration	ω Eq. (28)	Period, $\Delta\tau=2\pi/\omega$
0	...	1.000	6.28
0.25	0.982	0.974	6.40
0.5	0.911	0.896	6.90
0.75	0.785	0.766	8.00

We consider, then, Eq. (20) with $A=0$:

$$(\theta^4/12)u''=(1-\theta^2)u-\alpha\theta^3u^2-\eta\theta^2u'. \qquad (21)$$

This is the equation of a damped oscillator. As $\xi\rightarrow\infty$ it comes to equilibrium and the derivatives of u become arbitrarily small. We can then evaluate u_1 by the condition that, as $\xi\rightarrow\infty$,

$$u\rightarrow u_1,\ u'\rightarrow 0,\ u''\rightarrow 0. \qquad (22a)$$

Then, as before,

$$u_1=(1-\theta^2)/\alpha\theta^3. \qquad (22b)$$

In discussing Eq. (21), it is useful to make the transformation $u=u_1y$, $\xi=\theta^2\tau/[12(1-\theta^2)]^{1/2}$. Then Eq. (21) becomes

$$d^2y/d\tau^2=y-y^2-\eta[12/(1-\theta^2)]^{1/2}(dy/d\tau). \qquad (23)$$

This is the equation of a particle moving, with damping, in a potential,

$$\phi(y)=y^3/3-y^2/2, \qquad (24)$$

shown in Fig. 4. It has a maximum at $y=0$, a minimum at $y=1$, and it goes monotonically to infinity outside these limits. By virtue of the boundary conditions at $\xi\rightarrow-\infty$, viz. $u=u'=0$, the initial position of this "pseudoparticle" is at the maximum $\phi=0$, $y=0$. When the shock arrives, the pseudoparticle is moved to the right by the first infinitesimal disturbance and it

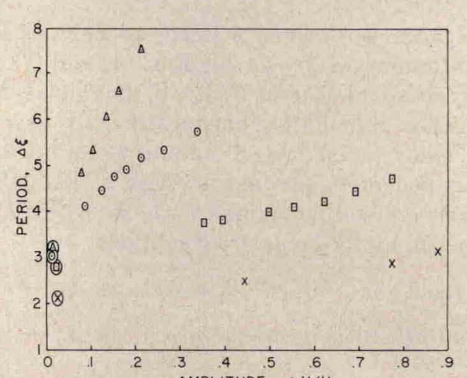

FIG. 5. Effects of $u_1\alpha$ and amplitude on period, Eq. (12): △—$\alpha=0.1$, $u_1=0.1$, $N=90$; ⊙—$\alpha=1.0$, $u_1=0.1$, $N=90$; ⊡—$\alpha=3.0$, $u_1=0.1$, $N=90$; ×—$\alpha=10.0$, $u_1=0.1$, $N=90$; for circled data points, $N=30$

accelerates, passes the minimum in ϕ, comes to rest at $y\leq\frac{3}{2}$, then returns toward its initial position, which it fails to reach because of dissipative forces, however small they may be. It then oscillates forever, unless it is overdamped, eventually coming to rest at $y=1$ to satisfy the boundary condition of Eq. (22) as $\xi\rightarrow\infty$.

As the amplitude of oscillation decays, the frequency of oscillation increases and the center of oscillation shifts toward larger values of y, i.e., toward $y=1$. Both these effects are consequences of the anharmonic potential of Eq. (24). The dependence of frequency on amplitude can be estimated from a perturbation calculation. Set $\eta=0$ in Eq. (23) and let $y=x+1$. Then Eq. (23) becomes

$$x''+x+x^2=0, \tag{25}$$

where $x'=dx/d\tau$, etc. Now if we let

$$x=x^{(1)}+x^{(2)}+x^{(3)}+\cdots$$

$$\omega=1+\omega^{(1)}+\omega^{(2)}+\cdots, \tag{26}$$

where successive terms are decreasing in magnitude, we find that[8]

$$x=-a^2/2+a\cos\omega\tau+(a^2/6)\cos2\omega\tau+(a^3/48)\cos3\omega\tau+\cdots \tag{27}$$

$$\omega=1-5a^2/12+\cdots. \tag{28}$$

For comparison, we have integrated Eq. (25) numerically for three values of a with the results shown in Table I. The two sets of results are comparable but far from identical. If a is corrected to yield the correct values of $x(0)=-0.25$, -0.50, -0.75, from Eq. (27), the agreement is somewhat improved. In any event the analysis confirms two points suggested by Fig. 4: The frequency of oscillation decreases and the center of oscillation moves to the left as the amplitude increases.

Equation (12) has been integrated numerically with the boundary condition $S_1'=u_1=$ constant for several values of α and u_1 with $\eta=0$. The variation of u_N with time has been determined for $N=30$, 60, and 90, and from each of these functions, which have the form shown in Fig. 1, the amplitude and period of oscillation have been determined as functions of time. Pertinent data for four integrations are given in Table II. Period, in units of ξ, is shown in Fig. 5 as a function of amplitude and $u_1\alpha$ for fixed N. The increase in amplitude with N and the dependence of period on N for fixed $u_1\alpha$ are shown in Fig. 6.

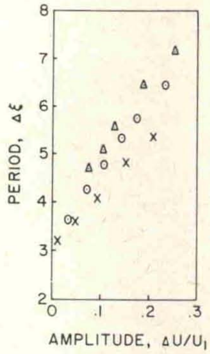

FIG. 6. Effects of travel distance (N) on amplitude and period: $\alpha=0.3$, $u_1=0.1$: ×—$N=30$; ⊙—$N=60$; △—$N=90$.

[8] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon Press, Inc., New York, 1960), Vol. 1 of Course of Theoretical Physics, p. 86.

V. DISCUSSION AND CONCLUSIONS

Numerical results from the transient problem [Eq. (12)] show that the peak amplitude of oscillation of the Nth particle increases with N for fixed α and with α for fixed N. The former result suggests that when N is very large, $u_{\min}/u_1\rightarrow0$, in accord with Eq. (23) and Fig. 4 and with the concept that the permanent regime solution for the lattice is a steady oscillation in which u returns periodically to zero. The prediction of Fig. 4 that the maximum value of u is $\frac{3}{2}u_1$ depends on truncation of the series in Eq. (16) and does not agree with the numerical integration. The increase of amplitude with α for fixed N is in accord with the continuum result that the rate of approach of a shock wave to its permanent regime profile increases with the curvature of the adiabat.

The variation of period with $(\theta/u_1\alpha)^{1/2}$ bears little relation to the result of Eq. (28) and Table I. The zero point period of $\omega=1$ corresponds, in units of ξ, to

$$\Delta\xi=2\pi(\theta/12u_1\alpha)^{1/2}.$$

The coefficient of 2π, shown in Table II, varies more than tenfold for the cases displayed in Fig. 5. Yet the value of $\Delta\xi$ shown there does not vary more than about 60% at $(u-u_1)/u_1\simeq0$, where Eq. (28) has greatest validity. In the next higher approximation to Eq. (16), the period for zero amplitude oscillations is

$$\Delta\xi=2\pi(\theta/12u_1\alpha)^{1/2}(1-2\alpha u_1\theta)^{1/2},$$

which varies even more rapidly with αu_1 than does the previous approximation. This result does indicate, however, that the variation of period with $u_1\alpha$ is sensitive to the truncation of Eq. (16); resolution of this point may depend upon exact solution of Eq. (15). The weight of the evidence presented here is that Eq. (12) is rather a bad approximation to Eq. (15) when $\eta=0$, though it does produce a profile without a discontinuity.

A rather remarkable suggestion which comes from numerical integration of Eq. (12) is that the solution is essentially independent of u_1 and α for fixed value of $u_1\alpha$. The individual factors were varied by a factor of ten while the product was held constant, yet values of u/u_1 in the solution differed by little more than numerical error.